Effective energy-momentum tensor of strong-field QED with unstable vacuum

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Abstract

We study the influence of a vacuum instability on the effective energy-momentum tensor (EMT) of QED, in the presence of a quasiconstant external electric field, by means of the relevant Green functions. In the case when the initial vacuum, $|0, in>$, differs essentially from the final vacuum, $|0, out>$, we find explicitly and compared both the vacuum average value of EMT, $<0, in|T_{\mu\nu}|0, in>$, and the matrix element, $<0, out|T_{\mu\nu}|0, in>$. In the course of the calculation we solve the problem of the special divergences connected with infinite time T of acting of the constant electric field. The EMT of pair created by an electric field from the initial vacuum is presented. The relations of the obtained expressions to the Euler-Heisenberg's effective action are established.

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1 Introduction

Currently, an effective action method, originating with Euler and Heisenberg's one-loop effective action, is one of the commonly used approaches of QFT. Nevertheless, we can see that if an external electric field is involved then naive calculations by analogy with a magnetic field case can be erroneous. For example, thermally influenced pair production in a constant electric field has been searched via several attempts of generalization for one-loop effective action at finite temperature with extremely contrary results. We would like to express that the vacuum instability in an electric background opens additional channels of interaction due to particles creation from the vacuum. That is the reason that

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the above mentioned simple analogy does not work, regardless of thermal influence, and we have to refine setting up a problem. For simplicity of explanation in this talk we suppose that the temperature is equal to zero[1].

The relevant intense field method, applicable to the theory with unstable vacuum in the case of a time-varying external field (called generalized Furry representation), can be found in a book [2][2]. Following this method we see that the effective perturbation theory with respect to the radiative interaction for the matrix elements of the scattering processes and another one for the expectation values differ by the type of the one-particle Green function due to the nontrivial difference between a final vacuum, $|0, out>$, and an initial vacuum, $|0, in>$, $c_v =< 0, out|0, in>$, $|c_v|^2 \neq 1$. Feynman diagrams for the matrix elements of the scattering processes have to be calculated by means of the causal propagator

$$S^c(x, x') = c_v^{-1} i < 0, out|T\psi(x)\bar{\psi}(x')|0, in>, \tag{1}$$

where $\psi(x)$ is a massive (m) quantum spinor field satisfying the Dirac equation with an external field. In the calculation of the expectation values one has to use the one-particle Green functions

$$\begin{aligned} S^c_{in}(x, x') &= i < 0, in|T\psi(x)\bar{\psi}(x')|0, in>, \\ S^c_{out}(x, x') &= i < 0, out|T\psi(x)\bar{\psi}(x')|0, out> . \end{aligned} \tag{2}$$

Both differ from the causal propagator (1). Additionally, these distinct Green functions are used to represent various matrix elements of operators of the current and energy-momentum tensor (EMT), and effective action beginning with zeroth order with respect to radiative interaction. Euler and Heisenberg's one-loop effective action Y_{out-in} is related to the causal propagator, $Y_{out-in} = i\mathrm{Tr}\ln S^c$. Varying the Y_{out-in}, given by the Fock-Schwinger proper time representation [1], one gets the following matrix elements of the operators of a current density, j_μ, and EMT, $T_{\mu\nu}$, in the one-loop approximation:

$$< j_\mu >^c =< 0, out|j_\mu|0, in> c_v^{-1} \,, \; < T_{\mu\nu} >^c =< 0, out|T_{\mu\nu}|0, in> c_v^{-1} \,, \tag{3}$$

where the operators j_μ and $T_{\mu\nu}$ are in the generalized Furry representation,

$$\begin{aligned} j_\mu &= \frac{q}{2}\left[\bar{\psi}(x), \gamma_\mu \, \psi(x)\right], \;\; T_{\mu\nu} = \frac{1}{2}\left(T^{can}_{\mu\nu} + T^{can}_{\nu\mu}\right), \\ T^{can}_{\mu\nu} &= \frac{1}{4}\left\{\left[\bar{\psi}(x), \gamma_\mu P_\nu \, \psi(x)\right] + \left[P^*_\nu\bar{\psi}(x), \gamma_\mu \, \psi(x)\right]\right\} \,, \\ P_\mu &= i\partial_\mu - qA_\mu(x), \;\; q = -e. \end{aligned} \tag{4}$$

On the other hand, at a time instant x^0 the average values of the j_μ and $T_{\mu\nu}$ operators in the one-loop approximation are the following

$$< j_\mu >^{in} =< 0, in|j_\mu|0, in>, \; < T_{\mu\nu} >^{in} =< 0, in|T_{\mu\nu}|0, in> . \tag{5}$$

[1]We will present the relevant generalization for one-loop effects at finite temperature anywhere.

[2]The extension of such an approach for finite temperature QED was presented in [3].

The equalities $< j_\mu >^{in} = < j_\mu >^c$ and $< T_{\mu\nu} >^{in} = < T_{\mu\nu} >^c$ hold strictly for theory with stable vacuum. Thus, the well known explicit expression of the Y_{out-in} [1] is useless for the calculation of any average values and we see it is desirable to find the relevant one-loop effective description for QED with a constant uniform electromagnetic field.

2 Proper time representation

To see the difference between S^c_{in} and S^c explicitly one can express these functions via the sets of the appropriate solutions of the Dirac equation in an external field (see details in [2]). First, we need two complete and orthonormal sets of the in/out-solutions of the Dirac equation, $\{_\pm\psi_n(x)\} / \{^\pm\psi_n(x)\}$. They describe particles (+) and antiparticles (−) at the initial/final time instant x^0_{in}/x^0_{out}. Second, we find decomposition coefficients $G\left({}_\zeta|^{\zeta'}\right)$ of the out-solutions in the in-solutions [3],

$$
{}^\zeta\psi(x) = {}_+\psi(x) G\left({}_+|^\zeta\right) + {}_-\psi(x) G\left({}_-|^\zeta\right) \ , \tag{6}
$$

Then from (1) we get the Feynman definition:

$$
\begin{aligned}
S^c(x,x') &= \theta(x_0 - x'_0) S^-(x,x') - \theta(x'_0 - x_0) S^+(x,x') \,, \\
S^-(x,x') &= i \sum_{n,m} {}^+\psi_n(x) \, G\left({}_+|^+\right)^{-1}_{nm} \, {}_+\bar\psi_m(x') \,, \\
S^+(x,x') &= i \sum_{n,m} {}_-\psi_n(x) \left[G\left({}_-|^-\right)^{-1} \right]^*_{nm} \, {}^-\bar\psi_m(x') \,,
\end{aligned} \tag{7}
$$

and for S^c_{in} we have

$$
\begin{aligned}
S^c_{in}(x,x') &= \theta(x_0 - x'_0) S^-_{in}(x,x') - \theta(x'_0 - x_0) S^+_{in}(x,x') \,, \\
S^\mp_{in}(x,x') &= i \sum_n {}_\pm\psi_n(x) {}_\pm \bar\psi_n(x') \,.
\end{aligned} \tag{8}
$$

Then one can express the difference as follows,

$$
\begin{aligned}
S^a(x,x') &= S^c(x,x') - S^c_{in}(x,x'), \\
S^a(x,x') &= -i \sum_{nm} {}_-\psi_n(x) \left[G({}_+|^-) G({}_-|^-)^{-1} \right]^\dagger_{nm} {}_+\bar\psi_m(x') \quad ,
\end{aligned} \tag{9}
$$

and the similar expression can be written for $S^p(x,x') = S^c(x,x') - S^c_{out}(x,x')$. Only if the vacuum is stable then all the coefficients $G({}_+|^-)$, and then S^a, S^p are equal to zero.

We consider the general case of a constant uniform electromagnetic field, $F_{\mu\nu}$, with nonzero invariants where an electric field is given by the time dependent

[3] We are using a convention of summation/integration over discrete/continuous repeated indices and a compact notation where all summations/integrations are supressed, for example $\psi_n G_{nm} = (\psi G)_m$. In addition ħ=c=1 throughout this paper.

potential. For simplicity, we choose the reference frame in which the electric, **E**, and magnetic, **B**, fields are parallel and directed along the x^3 axis.

All the singular functions in a constant uniform electromagnetic field can be represented [6] as the following Fock-Schwinger proper time integrals,

$$
\begin{aligned}
S^{c,a,p}(x,x') &= (\gamma P + m)\Delta^{c,a,p}(x,x'), \\
\Delta^{c}(x,x') &= \int_{\Gamma_c} f(x,x',s)ds = \int_0^{\infty} f(x,x',s)ds, \\
\Delta^{a/p}(x,x') &= \frac{1}{2}\Delta^{\Gamma_2}(x,x') + \Delta^{\bar{a}/\bar{p}}(x,x'), \\
\Delta^{\Gamma_2}(x,x') &= \int_{\Gamma_2} f(x,x',s)ds, \\
\Delta^{\bar{a}/\bar{p}}(x,x') &= \left[\Theta(\pm y_3) - \frac{1}{2}\right]\int_{\Gamma_2} f(x,x',s)ds + \int_{\Gamma_a} f(x,x',s)ds \\
&+ \Theta(\pm y_3)\int_{\Gamma_3 - \Gamma_a} f(x,x',s)ds\ , \quad y_3 = x_3 - x_3'
\end{aligned}
\tag{10}
$$

where $f(x,x',s)$ is the known Fock-Schwinger proper time kernel [1] and all the contours of the integrals are shown on Fig. 1. The contours Γ_c and Γ_1 are placed below the singular points on the real axis everywhere outside of the origin. Outside of the origin the kernel has only one singular point, $s_1 = -i\pi/eE$, on the complex region between the line of the contours $\Gamma_c - \Gamma_1$ and the line of the contours $\Gamma_a - \Gamma_3$.

By using these representations one can uniformly express all the matrix elements of the j_μ and $T_{\mu\nu}$ operators, as follows

$$
\begin{aligned}
&< j_\mu >^{in} = < j_\mu >^{c} - < j_\mu >^{a},\ < T_{\mu\nu} >^{in} = < T_{\mu\nu} >^{c} - < T_{\mu\nu} >^{a}, \\
&< j_\mu >^{out} = < j_\mu >^{c} - < j_\mu >^{p},\ < T_{\mu\nu} >^{out} = < T_{\mu\nu} >^{c} - < T_{\mu\nu} >^{p}, \\
&< j_\mu >^{c,a,p} = iq\ \mathrm{tr}_s\left\{\gamma_\mu\gamma^\nu P_\nu \Delta^{c,a,p}(x,x')\right\}\big|_{x=x'}\ , \\
&< T_{\mu\nu} >^{c,a,p} = i\ \mathrm{tr}_s\left\{B_{\mu\nu}\Delta^{c,a,p}(x,x')\right\}\big|_{x=x'}\ , \\
&\quad B_{\mu\nu} = 1/4\left\{\gamma_\mu\left(P_\nu + {P'}^{*}_{\nu}\right) + \gamma_\nu\left(P_\mu + {P'}^{*}_{\mu}\right)\right\}\gamma^\kappa P_\kappa\ , \\
&\quad P'^{*}_{\mu} = -i\frac{\partial}{\partial x'^{\mu}} - q\,A_\mu(x'),
\end{aligned}
\tag{11}
$$

where $\mathrm{tr}_s\{\ldots\}$ is the trace of an product of the Dirac gamma matrices.

The expression for the term $< j_\mu >^c$ in (11) is finite after the proper time regularization lifting and equal to zero. The components $< j_\mu >^{a/p}$ for $\mu \neq 3$ are equal to zero, as well. All the off-diagonal matrix elements of $< T_{\mu\nu} >^{c,a,p}$ are equal to zero. It is precisely the term $< T_{\mu\nu} >^c$ that can be derived from the Heisenberg-Euler effective Lagrangian, $\mathcal{L}$. Performing the standard renormalizations, leaving $eF_{\mu\nu}$ invariant, one gets the finite expression of the $< T_{\mu\nu} >^c$

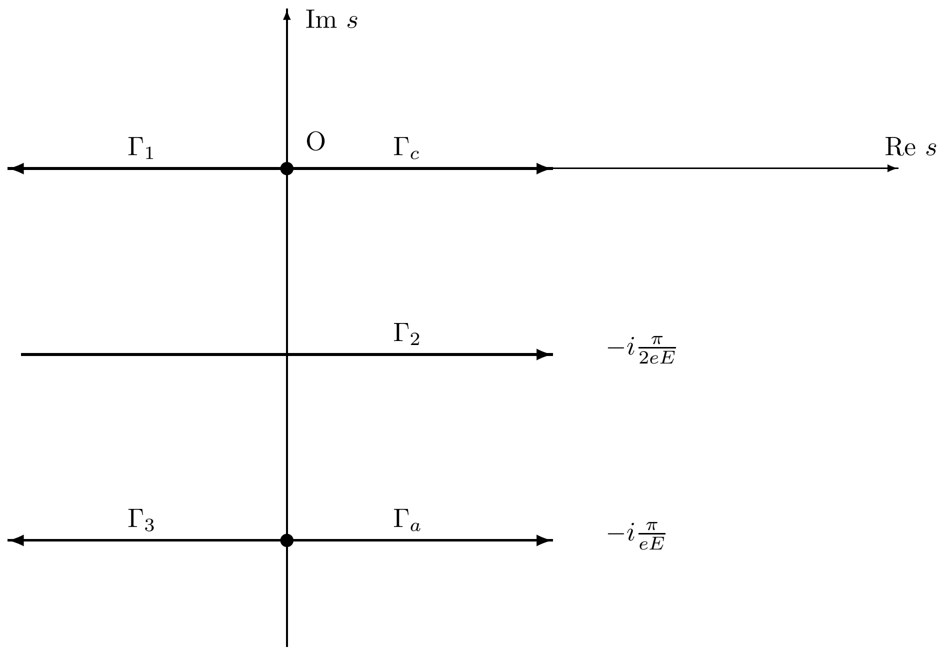

Figure 1: Contours of integration $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_c, \Gamma_a$.

as follows

$$< \ T_{00} >^c_{eff} = - < T_{33} >^c_{eff} = E\frac{\partial \mathcal{L}}{\partial E} - \mathcal{L}, \ \ < T_{11} >^c_{eff} = < T_{22} >^c_{eff} = \mathcal{L} - B\frac{\partial \mathcal{L}}{\partial B},$$
$$\mathcal{L} = \int_0^\infty \frac{ds}{8\pi^2 s} e^{-im^2 s} \left[e^2 EB \coth(eEs) \cot(eBs) - \frac{1}{s^2} - \frac{e^2}{3}\left(E^2 - B^2\right)\right].$$

Bearing in mind that $< T_{\mu\nu} >^{\Gamma_2} = 2i\mathrm{Im} < T_{\mu\nu} >^c$, we get for the average values of the operators j_μ and $T_{\mu\nu}$ the following explicitly real expressions,

$$< j_\mu >^{in} = - < j_\mu >^{\bar{a}}, \ < T_{\mu\nu} >^{in}_{eff} = \mathrm{Re} < T_{\mu\nu} >^c_{eff} - < T_{\mu\nu} >^{\bar{a}} . \tag{12}$$

The terms $< j_\mu >^{\bar{a}/p}$ and $< T_{\mu\nu} >^{\bar{a}/p}$ are proportional to the factor $\exp\left\{-\pi m^2/eE\right\}$. Thus, they are related to global features of the theory and indicate the vacuum instability. These matrix elements are free from the standard ultraviolet divergences. However, with such terms we run into special kind of divergences in the constant electric field due to the contributions from derivatives of $\Theta(\pm y_3)$ functions and singular point s_1. The nature of such special divergences is connected with infinite time T of acting of the constant electric field. They have to be regularized with respect to time T of acting of a constant electric field.

3 Finite work regularization

The state of the quantum system in question is far-from-equilibrium due to the influence of the time dependent potential of an electric field. Then there exists the problem of time dependence for average values which we discuss here. In a physically correct statement of the problem, we only refer to a quasiconstant electric field which is effectively acting during a finite time T, $E(x^0) = E$ for $t_1 \leq x^0 \leq t_2$, $t_2 = -t_1 = T/2$, and then does finite work in a finite volume. Out of the time interval T an electric field is absent. Further we call it a T-constant field. In this case the initial vacuum is the vacuum of free particles. General aspects of the special regularization with respect to time T by using the T-constant field was discussed in [4]. Now, we need to apply those results for calculating of the leading terms in $< j_3 >^{a/p}$ and $< T_{\mu\nu} >^{a/p}$at $T \to \infty$.

The mean number of particles created by the external field from the initial vacuum is

$$R_n^{cr} =< 0, in|a_n^\dagger(out)a_n(out)|0, in >= \left|G\left({}_-|^+\right)\right|^2, \tag{13}$$

were the standard volume regularization was used, so that $\delta(\mathbf{p}-\mathbf{p}') \to \delta_{\mathbf{p},\mathbf{p}'}$. If the time T is sufficiently large: $T >> T_0$, where $T_0 = (1+\lambda)/\sqrt{eE}$ is called the stabilization time, and $eET/2 \gg |p_3|$, then

$$\begin{aligned} R_n^{cr} &= e^{-\pi\lambda}\left[1 + O\left(\left[\frac{1+\lambda}{K}\right]^3\right)\right], \quad -\sqrt{eE}\frac{T}{2} \leq \xi \leq -K, \\ \lambda &= \frac{m^2 + \left\langle P_\perp^2 \right\rangle}{eE}, \; P_\perp = (P^1, P^2, 0), \; \xi = (|p_3| - eET/2)/\sqrt{eE}, \end{aligned} \tag{14}$$

where K is a sufficiently large arbitrary constant, $K >> 1+\lambda$, $\left\langle P_\perp^2 \right\rangle$ is the conserved average value of the $P_\perp^2$, and p_3 is a longitudinal momentum. The R_n^{cr} distribution for large longitudinal momenta, $|p_3| >> eET/2$, decreases, $R_n^{cr} = O\left(\left[\lambda/\xi^2\right]^3\right)$. The latter expression allows one to consider the limit $T \to \infty$ at any given quantum number. In this limit the distribution function takes the simple form $R_n^{cr} = e^{-\pi\lambda}$ which coincides with the expressions obtained in the constant electric field [5].

The distribution R_m^{cr} plays role of the cut-off factor for the integral (9) and similar representation of the S^p, then the contributions of $S^{a/p}$ are convergent. If the time interval $x^0 - t_1 = x^0 + T/2$ is sufficiently large, $\sqrt{eE}\left(x^0 + T/2\right) \gg 1 + m^2/eE$, we can extract the leading contributions at large T (marked a subscribt "as") in the representations (11), (12) and then, integrating over quantum numbers and calculating derivatives, obtain that

$$\begin{aligned} &< j_\mu >_{as}^{a/p} = -\delta_\mu^3 2e\left(1/2 \pm x^0/T\right) n^{cr}, \\ &< T_{00} >_{as}^{a/p} =< T_{33} >_{as}^{a/p} = -eET\left(1/2 \pm x^0/T\right)^2 n^{cr}, \\ &< T_{11} >_{as}^{a/p} =< T_{22} >_{as}^{a/p} \\ &= \widetilde{n}\begin{cases} \mp\ln\left[\sqrt{eE}\left(T/2 \pm x^0\right)\right] + O(\ln K) & \text{if } \sqrt{eE}\left(T/2 \pm x^0\right) > K \\ O(\ln K) & \text{if } \sqrt{eE}\left(T/2 \pm x^0\right) \leq K \end{cases} \end{aligned} \tag{15}$$

where K is an arbitrary constant, $K \gg 1 + m^2/eE$,

$$\begin{aligned} n^{cr} &= \frac{e^2 EBT}{4\pi^2} \coth \frac{\pi B}{E} \left[\exp\left\{ -\pi \frac{m^2}{eE} \right\} + O\left(\frac{K}{\sqrt{eET}} \right) \right], \\ \widetilde{n} &= \frac{e^2 B^2}{4\pi^2 \sinh^2(\pi B/E)} \exp\left\{ -\pi \frac{m^2}{eE} \right\}. \end{aligned} \tag{16}$$

Note that here n^{cr} is a characteristic number density of excitable states in the external field and, as we see subsequently, it is the same as the number density of the created pairs for time T of the duration of the electric field.

The current density and the EMT of the final particles created from vacuum by the T-constant field for the large time interval $x^0 - t_1 = x^0 + T/2 \gg K/\sqrt{eE}$ can be presented as

$$j_\mu^{cr} = < j_\mu >^{in} - < j_\mu >^{out}, \quad T_{\mu\nu}^{cr} = < T_{\mu\nu} >^{in} - < T_{\mu\nu} >^{out}, \tag{17}$$

where the terms $< j_\mu >^{out}$ and $< T_{\mu\nu} >^{out}$ are used to take into account the normal ordering of the current density and the EMT operators with respect to creation and annihilation operators of the final particles. Then one gets from (11) and (15) that

$$\begin{aligned} j_\mu^{cr} &= < j_\mu >_{as}^{p} - < j_\mu >_{as}^{a} = \delta_\mu^3 2e \left(2x^0/T \right) n^{cr}, \\ T_{\mu\nu}^{cr} &= < T_{\mu\nu} >_{as}^{p} - < T_{\mu\nu} >_{as}^{a}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} < & T_{00} >^{cr} = < T_{33} >^{cr} = 2eEx^0 n^{cr}, \\ < & T_{11} >^{cr} = < T_{22} >^{cr} \\ = & \ \widetilde{n} \begin{cases} \ln\left[eE \left((T/2)^2 - \left(x^0\right)^2 \right) \right] + O(\ln K) \ \text{if} \ \sqrt{eE}\left(T/2 - x^0\right) > K \\ \ln\left[\sqrt{eE}\left(T/2 + x^0\right) \right] + O(\ln K) \ \text{if} \ \sqrt{eE}\left(T/2 - x^0\right) \le K \end{cases} . \end{aligned} \tag{19}$$

At $x_0 = t_2 = T/2$ one gets from (18), (19) the expressions for the total current densities and the EMT of the particles created.

4 Conclusion

We finally obtain the average values of the current density and the EMT as following

$$< j_\mu >^{in} = - < j_\mu >_{as}^{a}, \ < T_{\mu\nu} >_{eff}^{in} = \mathrm{Re} < T_{\mu\nu} >_{eff}^{c} - < T_{\mu\nu} >_{as}^{a}. \tag{20}$$

As we have seen, the T dependent contributions to $< j_\mu >^{\bar{a}}$ and $< T_{\mu\nu} >^{\bar{a}}$ appear due to the vacuum instability and then come with the factor $\exp\left\{-\pi m^2/eE\right\}$. This factor is exponentially small for a weak electric field, $m^2/eE \gg 1$, and the effect can actually be observed as soon as the external field strength approaches

the characteristic value $E_c = m^2/e$. On the other hand, the term Re $< T_{\mu\nu} >^c_{eff}$ is the T independent and its contribution is not small whether the electric field is weak or strong. When the T-constant electric field is switched off at $x^0 > T/2$, the local vacuum contribution of the E in the Re $< T_{\mu\nu} >^c_{eff}$ is absent but the global contribution given by the $< T_{\mu\nu} >^a_{as}|_{x^0=T/2}$ is presence. Thus, in general case both kinds of contributions are important.

Using the expression (20) we find a condition for validity of a strong constant electric field concept. With a very strong E field, $m^2/eE \ll 1$ ($B = 0$), and large T one gets the well known asymptotic expression of the Re $< T_{00} >^c_{eff}$ vacuum energy density,

$$\mathrm{Re} < T_{00} >^c_{eff} = -\frac{e^2}{24\pi^2} E^2 \ln \frac{eE}{m^2}.$$

It is T independent contribution. The energy density of a classic electric field is $E^2/8\pi$. Then it seems that an electric field concept is physically meaningful when $\frac{e^2}{3\pi} \ln \frac{eE}{m^2} \ll 1$. But when T is large one has to give attention to the T dependent term $< T_{00} >^a_{as}$. At $x^0 = T/2$ we have

$$- < T_{00} >^a_{as} = \frac{e^2 E^2}{4\pi^3} eET^2.$$

Of course, one can neglect a back-reaction on an electric field only if the last term is far less than $E^2/8\pi$. Thus, the true condition for validity of a strong constant electric field concept is the following

$$1 \ll eET^2 \ll \frac{\pi^2}{2e^2}.$$

All the results for a pair creation are valid within the accuracy of the analysis at low density and temperature, $\Theta \ll eET$.

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